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Renormalization in quantum field theory: an improved rigorous method

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Abstract

The perturbative construction of the S-matrix in the causal spacetime approach of Epstein and Glaser may be interpreted as a method of regularization for divergent Feynman diagrams. The results of any method of regularization must be equivalent to those obtained from the Epstein–Glaser (EG) construction, within the freedom left by the latter. In particular, the conceptually well-defined approach of Bogoliubov, Parasiuk, Hepp and Zimmermann (BPHZ), though conceptually different from EG, meets this requirement. Based on this equivalence we propose a *modified* BPHZ *procedure* which provides a significant simplification of the techniques of perturbation theory, and which applies equally well to standard quantum field theory and to chiral theories. We illustrate the proposed method by a number of examples of various orders in perturbation theory. At the level of multi-loop diagrams we confirm that subdiagrams as classified by Zimmermann's forest formula in BPHZ can be restricted to subdiagrams in the sense of Epstein–Glaser, thus entailing an important reduction of actual computations. The relationship of our approach to the method of dimensional regularization (and renormalization) is particularly transparent, without having to invoke analytic continuation to unphysical spacetime dimension. It sheds new light on the role of some parameters that appear within the dimensional regularization, and thus establishes a direct link of this traditional method to the BPHZ scheme.

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1. Introduction

As it is well known, formal perturbation theory applied to relativistic quantum field theory, in general, leads to ill-defined expressions for the elements of the S-matrix. Integrals that are expected to describe probability amplitudes for certain scattering processes are found to be divergent. Generally speaking, these ultraviolet (UV) divergencies can be traced back to the naïve application of time ordering in the description of propagation of particles. There are two conceptually rather different lines of attack to deal with this problem: the first of these consists of a set of *regularization* procedures, all of which are designed to replace divergent integrals in Feynman diagrams by convergent ones in a consistent manner. These empirical regularization schemes are justified by their usefulness in practical applications of quantum field theory to physical processes. In order to be consistent, they must fulfill all physical (normalization) conditions, order by order, or, at least, must contain enough freedom to meet these conditions. This is the essential prerequisite for the procedure of renormalization. In other terms, not every scheme of *regularization* of divergent integrals of a given theory meets the stronger requirement of *renormalizability* of that theory.

The second line follows the approach developed by Epstein and Glaser, [1], which is based on causality and locality in coordinate space. This procedure makes use of a well-defined rule for time ordering and thereby allows us to construct an entirely divergence-free S-matrix from basic and general principles. The Epstein–Glaser (EG) approach is mathematically rigorous, within perturbation theory, but, when applied without modifications, is not very useful in practice. By its very construction, due to the process of distribution splitting, it contains a certain freedom which, subsequently, is fixed through its interpretation in terms of physics in the process of renormalization. In fact, as it was first proposed in [2, 3], the EG method can be interpreted itself as a regularization scheme. Thus, this approach is particularly useful as a reference framework for testing whether a given empirical method of regularization is physically admissible (in the sense of renormalizability), or not.

Among the regularization procedures of the first group the classical method of Bogoliubov, Parasiuk, Hepp and Zimmermann (BPHZ), by its conceptual clarity, is the most rigorous [4, 5]. The rules of regularization that it contains at the level of one-loop diagrams are equivalent to those of EG, with the latter suitably translated to momentum space. In addition BPHZ contains a general prescription, in the form of the *forest formula*, for regularizing higher loop diagrams. However, although its logical structure is transparent, the BPHZ procedure leads to the rather involved integrals in explicit calculations which make it less suitable for practical computations of Feynman amplitudes as compared to more empirical methods such as dimensional regularization or the like. (In using the term 'dimensional regularization' we follow common conventions. In fact, this nomenclature means dimensional renormalization with, say, minimal subtraction.)

With $d(x)$ a scalar distribution of singular order ω the EG method defines advanced and retarded distributions through splitting of its support by a space-like hypersurface, say $v\cdot x = 0$ with v a timelike vector. As it is well known, a construction valid for all physically relevant values ω and for all test functions $g \in \mathfrak{S}'(\mathbb{R}^k)$ is then

$$\int \mathrm{d}^4x\, d_{\mathrm{ret,reg}}(x) g(x) = \int \mathrm{d}^4x\, d(x)\Theta(v\cdot x)(Wg)(x), \tag{1a}$$

$$\int \mathrm{d}^4x\, d_{\mathrm{adv,reg}}(x) g(x) = -\int \mathrm{d}^4x\, d(x)[1-\Theta(v\cdot x)](Wg)(x), \tag{1b}$$

where the operator W is defined through its action on test functions $g(x)$:

$$(Wg)(x) = g(x) - w(x)\sum_{|a|=0}^{\omega} \frac{x^a}{a!}(D^a g)(0), \tag{2a}$$

with D^a the customary short-hand for partial derivatives

$$D^a = \frac{\partial^{a_1+\cdots+a_k}}{\partial x_1^{a_1}\cdots\partial x_k^{a_k}}, \qquad |a| = a_1+\cdots+a_k,$$

with x^a the standard multicomponent notation for the coordinates and with $w(x)$ a function satisfying the conditions

$$w(0) = 1, \qquad D^a w|_{x=0} = 0 \qquad \text{for all} \quad 1 \leqslant |a| \leqslant \omega. \tag{2b}$$

A given choice of the function w represents a specific regularization. This is the perspective adopted in [2, 3]. Any two different regularizations differ by a δ-distribution and derivatives thereof up to the order ω in the integrands, [6], namely

$$\int \mathrm{d}^4 x (d_{\mathrm{reg},w_1}(x) - d_{\mathrm{reg},w_2}(x)) g(x) = \int \mathrm{d}^4 x \left(\sum_{|a|=0}^{\omega} c_a D^a \delta(x) \right) g(x) \quad \text{with}$$

$$c_a = \int \mathrm{d}^4 x\, d(x)(-1)^{|a|} \frac{x^a}{a!} [w_2(x) - w_1(x)].$$

This freedom of choice is essential for the subsequent renormalization process which relates the free parameters to the values of physical parameters. As also shown in that work, a modified subtraction operator such as the one proposed in [7] will yield a valid regularization but may turn out to be too restrictive for successful renormalization.

In this paper we study a new method which we propose to call the *modified* BPHZ *procedure*. This method combines the practical usefulness of dimensional regularization with the structural simplicity of the classical BPHZ renormalization in the light of its equivalence to the EG method. Similar to any other empirical regularization method, the rigorous EG framework is the landmark with respect to which the correctness and use of our modified procedure must be judged.

The paper is organized as follows. In section 2 we discuss the equivalence between the BPHZ and EG frameworks. In section 3 we describe the idea of the modified BPHZ method and its justification, by means of its relationship to EG. In section 4 we give some instructive examples and work out the relationship to the dimensional regularization. The final section 5 gives a summary and outlook.

2. Equivalence of BPHZ and EG frameworks

The BPHZ scheme is based on Feynman rules in momentum space. Schematically, and at this point still somewhat formally, a given diagram γ with internal momenta k is translated to an integrand of the form

$$I_\gamma(p,k) = \prod_{l \in \mathcal{L}} \Delta_c(p,k) \prod_{V \in \mathcal{V}} P_V(p,k), \tag{3}$$

where p denotes the set of external momenta, while k stands for the internal momenta to be integrated over. The factors Δ_c are proportional to Feynman propagators $\widetilde{\Delta}_F$ in momentum space, and correspond to the internal lines l of a given set $\mathcal{L}$. The momentum flow is defined by the conventions chosen in the forest formula. A given vertex V in the set $\mathcal{V}$ of vertices contributes the factor P_V. Consider an arbitrary irreducible one-loop diagram whose degree of divergence is $d(\gamma)$. The BPHZ approach replaces the integrand by the modified expression

$$R_\gamma(p,k) = \left(1 - t_p^{d(\gamma)}\right) I_\gamma(p,k), \tag{4a}$$

where

$$t_p^{d(\gamma)} = \sum_{|n|=0}^{d(\gamma)} \frac{1}{n!} p^n \left. \frac{\mathrm{d}}{\mathrm{d}p^n} \right|_{p=0}. \tag{4b}$$

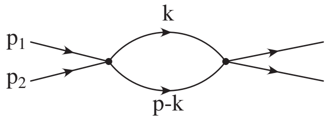

Figure 1. Four-point function in the ϕ^4 model, at one loop, with external momentum $p = p_1 + p_2$.

The Taylor operator $t_p^{d(\gamma)}$ stands symbolically for the expansion in terms of the set of independent external momenta p. In view of subsequent renormalization, the general result of regularization has the form

$$\int \mathrm{d}^4k\, I_\gamma(p,k)\Big|_{\mathrm{BPHZ,reg}} = \int \mathrm{d}^4k\, R_\gamma(p,k) + P^{(d(\gamma))}(p), \tag{5}$$

with $P^{(d(\gamma))}(p)$ a polynomial of degree $d(\gamma)$ representing the remaining freedom.

For the sake of illustration, we will refer repeatedly to ϕ^4 theory in which case P_V yields a power of the coupling constant g. As an example, consider the four-point function of ϕ^4 theory at one loop, i.e. the diagram shown in figure 1, with external momentum p and containing two internal lines. In this case $d(\gamma) = 0$ and BPHZ regularization yields

$$\widetilde{\Delta}_{\mathrm{F}}^2(p)|_{\mathrm{BPHZ,reg}} = -\frac{1}{(2\pi)^6}\int \mathrm{d}^4k\, \left\{\frac{1}{k^2-m^2}\frac{1}{(p-k)^2-m^2} - \frac{1}{(k^2-m^2)^2}\right\}. \tag{6}$$

In order to compare with the corresponding result of EG regularization, equations (1*a*) and (2*a*), as well as all test functions, are transformed to momentum space, by Fourier transform, so as to obtain

$$d_{\mathrm{reg}}(g) = \int \mathrm{d}^4x\, d_{\mathrm{reg}}(x)g(x) = \int \mathrm{d}^4k\, \widetilde{d}_{\mathrm{reg}}(k)\widetilde{g}(k), \tag{7}$$

as well as the analogue of (6),

$$\widetilde{\Delta}_{\mathrm{F}}^2(p)|_{\mathrm{reg}} = -\frac{1}{(2\pi)^6}\int \mathrm{d}^4k\, \frac{1}{k^2-m^2}\left\{\frac{1}{(p-k)^2-m^2} - \frac{1}{(2\pi)^2}\int \mathrm{d}^4p'\, \frac{\widetilde{w}(p')}{(p'-k)^2-m^2}\right\}. \tag{8}$$

In the present example one may choose the function w to be $w(x) = 1$, as a limiting case. Obviously, it satisfies the conditions (2*b*). Furthermore, its Fourier transform being $\tilde{w}(p) = (2\pi)^2\delta(p)$, is seen to yield the BPHZ expression (6).

Of course, proving the equivalence of the BPHZ and EG schemes, beyond the one-loop level and for other theories, becomes technically more complicated. As it is well known, this is due to the fact that EG is an expansion in terms of the number n of vertices, i.e. in terms of powers of the coupling constant, while BPHZ is an expansion in terms of loops, i.e. a formal expansion in terms of Planck's constant.

3. A modified BPHZ procedure

In the classical BPHZ method divergent momentum integrals are regularized by means of appropriate Taylor subtractions of the integrand. Although, on the basis of Zimmermann's forest formula, this approach is transparent and well defined in principle, its practical implementation in higher orders of the perturbation theory is cumbersome. From a practical point of view, other methods of regularization, such as analytic continuation in the dimension of spacetime, are better tools in actual calculations.

The alternative procedure that we propose aims at modifying the well-defined framework of BPHZ in such a way that it becomes as practicable as customary dimensional regularization. In essence, the idea is to introduce Feynman parameters at the level of the *unsubtracted* integrand and to apply Taylor subtraction to the modified integrand only. We will show this preserves the mathematical rigour of the BPHZ scheme but simplifies enormously subsequent integrations over internal momenta.

To start with, and in order to explain the essence of the modified method, we give a very simple example from the ϕ^4 model. Within the BPHZ scheme and at second order in the coupling constant g, the contribution of the one-loop diagram to the four-point function is logarithmically divergent. BPHZ regularize it by Taylor subtraction of the integrand to order zero, namely

$$\frac{1}{2}g^2\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\big(1-t_p^0\big)\frac{1}{[k^2-m^2]}\frac{1}{[(p-k)^2-m^2]} =: \Lambda(p), \tag{9}$$

and thus obtain a well-defined expression. In a first step, and in analogy to dimensional regularization, we parametrize the unmodified integrand by means of a Feynman parameter z, such that the integrations over z and over the internal momentum k may be interchanged. In a second step the integration variable is subject to a translation by the vector $(z-1)p$, $k \mapsto q = k + (z-1)p$, so that mixed terms containing external and internal momenta no longer appear. Finally, the Taylor subtraction is applied to the modified integrand. The three steps are given by, respectively,

$$\begin{aligned}\Lambda(p) &= \frac{g^2}{2(2\pi)^4}\int \mathrm{d}^4k\big(1-t_p^0\big)\int_0^1 \mathrm{d}z\frac{1}{\{[(p-k)^2-m^2](1-z)+z[k^2-m^2]\}^2}\\ &= \frac{g^2}{2(2\pi)^4}\int_0^1 \mathrm{d}z\int \mathrm{d}^4k\big(1-t_p^0\big)\frac{1}{[k^2-2pk(1-z)+p^2(1-z)-m^2]^2}\\ &= \frac{g^2}{2(2\pi)^4}\int_0^1 \mathrm{d}z\int \mathrm{d}^4q\big(1-t_p^0\big)\frac{1}{[q^2+z(1-z)p^2-m^2]^2}.\end{aligned} \tag{10}$$

Note that the scaling behavior of the integrand for large values of k remains unchanged by the introduction of a Feynman parameter. Therefore, the coefficients of the Taylor expansion of order higher than the degree of divergency lead to convergent integrals. The calculation of the integral over q is standard. Making use of a Wick rotation, one obtains

$$\Lambda(p) = \frac{\mathrm{i}g^2}{32\pi^2}\int_0^1 \mathrm{d}z \ln\left(\frac{m^2}{m^2-z(1-z)p^2}\right). \tag{11a}$$

Note that, unlike in dimensional regularization, result (11*a*) is exclusively obtained in dimension 4 of physical spacetime. Furthermore, the remaining freedom in the approach discussed here, which allows for a constant additive term (with respect to p), may be made explicit by replacing $m \mapsto \mu$, with μ an arbitrary mass, in the numerator of the logarithm in (11*a*), i.e. $\Lambda(p)$ may be replaced by

$$\Lambda^{(\mu)}(p) = \frac{\mathrm{i}g^2}{32\pi^2}\int_0^1 \mathrm{d}z\ \ln\left(\frac{\mu^2}{m^2-z(1-z)p^2}\right). \tag{11b}$$

Indeed, expressions (11*a*) and (11*b*) differ by a constant only. For instance, the specific choice $\mu^2 = 4\pi\mu^2_{\mathrm{dim.reg}}\,\mathrm{e}^{-\gamma}$, with γ Euler's constant, reproduces the well-known result of dimensional regularization,

$$\Lambda^{(\mathrm{dim.reg})}(p) = \frac{\mathrm{i}g^2}{32\pi^2}\left\{-\gamma + \int_0^1 \mathrm{d}z\ \ln\left(\frac{4\pi\mu^2_{\mathrm{dim.reg}}}{m^2-z(1-z)p^2}\right)\right\}. \tag{11c}$$

Somewhat more generally, a convergent one-loop integral whose integrand was Taylor subtracted to the appropriate order ω,

$$J_\gamma(p) = \int \mathrm{d}^4k\big(1 - t_p^\omega\big)I_\gamma(k, p), \tag{12a}$$

is transformed by a translation of the argument, $k \mapsto q = k + \lambda p$,

$$J_\gamma(p) = \int \mathrm{d}^4k\big\{\big(1 - t_p^\omega\big)I_\gamma(k + \lambda p, p) - \Delta(q, p)\big\}, \tag{12b}$$

where the function $\Delta(q, p)$ is defined by this equation and, hence, is given by

$$\Delta(q = k + \lambda p, p) = \big(1 - t_p^\omega\big)I_\gamma(k + \lambda p, p) - \big(1 - t_p^\omega\big)I_\gamma(q, p)|_{q=k+\lambda p}. \tag{13a}$$

Note that in the first term the Taylor operator t_p^ω applies to both arguments of the function I_γ, while in the second term it applies to the second argument only.

Alternatively, the function $\Delta(q, p)$, with $q = k + \lambda p$ can also be written as follows:

$$\Delta(q = k + \lambda p, p) = \big(1 - t_p^\omega\big)\big[\big(t_p^\omega I_\gamma(q, p)\big)\big|_{q=k+\lambda p}\big]. \tag{13b}$$

The result (13*b*) follows from an identity for the Taylor operator t_p^ω in the variable p about the point $p = 0$, applied to a differentiable function F of two variables,

$$t_p^\omega F(k + \lambda p, p) = t_p^\omega\big(t_p^\omega F(q, p)\big|_{q=k+\lambda p}\big). \tag{14}$$

Indeed, denoting by ∂_1 and ∂_2 the derivatives with respect to the first and second argument of F, respectively, the left-hand side is

$$t_y^N F(x + \lambda y, y) = \sum_{i=0}^{N} \binom{i}{N} \lambda^i y^i \big(\partial_1^i \partial_2^{N-i} F(x, 0)\big).$$

The right-hand side, in turn, is computed to be

$$\begin{aligned} t_y^N\big(t_y^N F(u, y)\big|_{u=x+\lambda y}\big) &= t_y^N \left\{ \sum_{k=0}^{N} \frac{1}{k!} y^k \big(\partial_2^k F(u, y)\big)_{u=x+\lambda y, y=0} \right\} \\ &= \sum_{i=0}^{N} \binom{i}{N} \sum_{k=0}^{N} \frac{1}{k!} \lambda^i y^i \big(\partial_1^i \partial_2^k F(x, 0)\big)\big(\partial_y^{N-i} y^k\big)\big|_{y=0}. \end{aligned}$$

With $\big(\partial_y^{N-i} y^k\big)\big|_{y=0} = k!\delta_{N-i,k}$, this is seen to be the same expression as above.

As a result, the integral J_γ, after translation of the internal momentum, takes the form

$$J_\gamma(p) = \int \mathrm{d}^4k\big(1 - t_p^\omega\big)I_\gamma(k + \lambda p, p) + J^{(0)}(p), \tag{15}$$

where $J^{(0)}$ is the integral over Δ. Closer examination of (13*a*) shows that this integrand can be written as a sum of derivatives with respect to k of order one and higher, and, hence, gives rise to surface terms which vanish at infinity. Thus, $J^{(0)}$ vanishes. This calculation demonstrates that the translation of the integration variable is an admissible operation.

As will be clear from the examples worked out below, the parameter λ, in general, is a function of the Feynman parameter(s) z. As in the example above, the translation is chosen such that the mixed terms in k and p disappear. The integrand is then a function of k^2 only so that the integration can be done in Euclidean polar coordinates, via Wick rotation.

These examples motivate the following modified BPHZ procedure.

(1) In a given integral $I_\gamma(k, p)$ with external and internal momenta p and k, respectively, introduce integral representations by means of a set of Feynman parameters z, interchange the z integrations with the operator $\big(1 - t_p^\omega\big)$ and with the integration over the internal momenta k.

(2) Perform a translation of the k-variables such that internal and external momenta are decoupled.
(3) To the integrand, apply Taylor subtraction up to singular order ω.
(4) Do the k-integrals by means of Wick rotation and using Euclidean polar coordinates.
(5) In order to make contact with dimensional regularization replace the mass parameter(s) by general constant(s) μ so that only additive terms appear which form a polynomial in p up to and including the singular order ω. In some cases this does not exhaust the freedom necessary for renormalization because, obviously, the modified BPHZ method has the same number of parameters as the original one. This is essential for identifying the physical parameters of the theory (masses, charges, etc) in each order of perturbation theory.

This is a well-defined algorithm whose advantages are evident. The general analysis given in equations (12*a*)–(15) as well as the examples at second and higher orders, lend strong support to the conjecture that it meets all requirements of physical renormalization. Its closeness to the original BPHZ regularization and, hence, to EG regularization, guarantees that it is an admissible regularization scheme.

By a suitable choice of the parameter(s) μ, one makes contact with well-known regularization methods such as dimensional regularization, without having to continue to unphysical spacetime dimensions. The method is mathematically rigorous but more practicable than the original BPHZ approach.

Furthermore, turning to fermions, no continuation of the Clifford algebra of Dirac γ-matrices is necessary given the fact that the modified BPHZ method works exclusively in dimension four.

Our approach is rather close to the framework of Epstein and Glaser, but allows for a direct comparison with unmodified BPHZ. Due to cancellations of a certain class of subdiagrams there are important simplifications in the calculation of higher order processes. In order to explain this point we need some preparation and definitions.

As we stated above, EG is an expansion in terms of the coupling constant g, while BPHZ is an expansion in powers of $\hbar$, hence in terms of the number of loops. EG constructs a functional T_n describing a diagram with n vertices by recurrence from the tempered distribution $T_1 = i\mathcal{L}_{\text{int}}$. The total diagram depends on functionals which were regularized previously at orders lower than n, say $m < n$. Thus, the corresponding subdiagrams contain irreducible parts with a number of vertices *smaller* than n. We shall call such subdiagrams EG subdiagrams for short. The BPHZ framework, in turn, works by successive addition of counter terms proportional to ascending powers of $\hbar$ and, as a consequence, requires a different classification of subdiagrams. Let us call the *BPHZ subdiagram* any irreducible divergent part of the total diagram which contains a smaller number of *loops* than the main diagram. In particular, there will be subdiagrams which are lower in loop order but do not have a smaller number of vertices. We call these *pure* BPHZ subdiagrams. An example we shall study in more detail below is the 'sunrise' diagram in the ϕ^4 model, cf figure 2. In the framework of BPHZ, it contains three logarithmically divergent subdiagrams. In the perspective of EG, in contrast, it is a diagram with two vertices and, hence, contains no divergent subdiagram at all. In our terminology the three BPHZ subdiagrams are *pure* BPHZ subdiagrams. The sum of the counter terms generated by these subdiagrams does not contribute to the regularization of the sunrise diagram. This example, as well as other examples studied in [6], confirms this to be a general rule, and are in accordance with a theorem by Zimmermann [10]: pure BPHZ subdiagrams do not yield counter terms, i.e. their sum vanishes, and, thus, they may be left out in the modified approach.

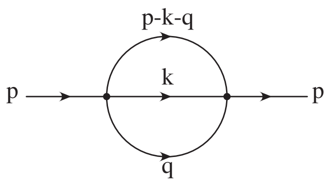

Figure 2. Sunrise diagram in the ϕ^4 model

We illustrate the method by a number of significant examples in second and higher orders.

4. Examples

We start with some classical examples from quantum electrodynamics and electroweak interactions, the self-energy of the electron, the vacuum polarization and the vertex correction at one-loop order, then mention briefly the case of the triangle anomaly. We finish with a typical second-order, two-loop process and with some remarks about higher order processes, which illustrate the simplicity of our alternative scheme. In all these examples the equivalence to EG regularization proves the correctness of the modified BPHZ approach.

4.1. Quantum electrodynamics with electrons

In the original BPHZ framework, the *self-energy* of the electron reads

$$\Sigma(p) = -\frac{\mathrm{i}e^2}{(2\pi)^4}\int \mathrm{d}^4k\big(1-t_p^1\big)\frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma^\mu}{[(p-k)^2-m^2]k^2}. \qquad (16a)$$

In the modified BPHZ approach, we introduce a Feynman parameter z, interchange integrations and substitute $k \mapsto q = k - zp$ so as to decouple internal and external momenta, to obtain

$$\Sigma(p) = -\frac{\mathrm{i}e^2}{(2\pi)^4}\int_0^1 \mathrm{d}z\int \mathrm{d}^4q\big(1-t_p^1\big)\frac{\gamma_\mu((1-z)\not{p}-\not{q}+m)\gamma^\mu}{[q^2-zm^2+z(1-z)p^2]^2}. \qquad (16b)$$

This is easily worked out to be

$$\Sigma(p) = \frac{e^2}{16\pi^2}\int_0^1 \mathrm{d}z[(z-1)2\not{p}+4m]\,\ln\left(\frac{m^2}{m^2-(1-z)p^2}\right). \qquad (16c)$$

The remaining freedom is made explicit by replacing m^2 by an arbitrary squared mass μ^2,

$$\Sigma^{(\mu)}(p) = \frac{e^2}{16\pi^2}\int_0^1 \mathrm{d}z[(z-1)2\not{p}+4m]\,\ln\left(\frac{\mu^2}{m^2-(1-z)p^2}\right). \qquad (16d)$$

One verifies that the choice

$$\mu^2 = 4\pi\mu^2_{\text{dim.reg}}\,\mathrm{e}^{(1/2-\gamma)} \qquad (17)$$

reproduces (the finite part of) the result known from dimensional regularization, see e.g. [11].

Lowest order *vacuum polarization* in the original BPHZ is given by

$$\Pi_{\mu\nu}(p) = \frac{\mathrm{i}e^2}{(4\pi)^2}\int \mathrm{d}^4q\big(1-t_p^2\big)\mathrm{tr}\,\left(\gamma_\mu\frac{\not{q}+m}{q^2-m^2}\gamma_\nu\frac{\not{q}-\not{p}+m}{(q-p)^2-m^2}\right). \qquad (18a)$$

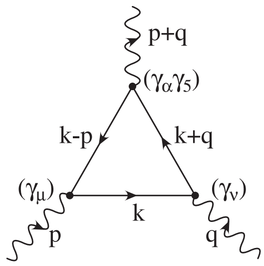

Figure 3. Triangle graph contributing to the anomaly.

In the modified scheme, we introduce a Feynman parameter z, interchange the integration over z with the one over the internal momentum q and perform a translation of the integration variable $q \mapsto \bar{q} = q - (1-z)p$ to obtain

$$\Pi^{(\mu)}_{\mu\nu}(p) = -\frac{e^2}{2\pi^2}\int_0^1 \mathrm{d}z(g_{\mu\nu}p^2 - p_\mu p_\nu)z(1-z)\ln\left(\frac{\mu^2}{m^2 - z(1-z)p^2}\right). \tag{18b}$$

As before, in order to exhaust the remaining freedom, we have replaced the numerator m^2 in the logarithm by an arbitrary squared mass μ^2. The (finite part of) the known result of dimensional regularization [11] is recovered by the choice

$$\mu^2 = 4\pi\mu^2_{\text{dim.reg}}\,\mathrm{e}^{-\gamma}. \tag{19}$$

The *vertex correction,* at the same order, finally, is found to be

$$\begin{aligned}-\mathrm{i}e\Lambda^{(\mu)}_\alpha(p,p') = {}& -\frac{\mathrm{i}e^3}{8\pi^2}\gamma_\alpha\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\ \ln\left(\frac{\mu^2(x+y)}{D^2}\right) + \frac{\mathrm{i}e^3}{8\pi^2}\gamma_\alpha + \frac{\mathrm{i}e^3}{16\pi^2}\\ &\times\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\ \frac{1}{D^2}\{\gamma_\nu[(1-y)\not{p}' - x\not{p} + m]\gamma_\alpha[(1-x)\not{p} - y\not{p}' + m]\gamma^\nu\},\end{aligned} \tag{20}$$

where the denominator in the integrands stands for

$$D^2 = (x+y)m^2 - x(1-x)p^2 - y(1-y)p'^2 + 2pp'xy.$$

As before, we replaced the numerator m^2 in the logarithm by an arbitrary term μ^2, to cope with the remaining freedom after regularization. The analogous result in dimensional regularization is recovered by the same choice (17) as for the self-energy. This shows that the modified BPHZ regularization fulfills the Ward–Takahashi identity

$$\frac{\partial}{\partial p^\alpha}\Sigma(p) = -\Lambda_\alpha(p,p). \tag{21}$$

Though not surprising, this is a consistency check.

4.2. Chiral anomaly

We also analyzed the well-known vector–vector–axial vector (VVA) chiral anomaly shown in figure 3, within the modified BPHZ procedure.

Denoting the amplitude by $T_{\alpha\mu\nu}$ and choosing the internal loop momenta as shown in figure 3, conservation of the vector current at the two lower vertices should yield the Ward identities

$$p^\mu T_{\alpha\mu\nu} = 0, \qquad q^\nu T_{\alpha\mu\nu} = 0, \tag{22a}$$

whereas the axial current vertex should produce an anomalous Ward identity which survives even in the limit of the fermion mass m going to zero, namely

$$(p+q)^\alpha T_{\alpha\mu\nu} = 2mT_{\mu\nu} + \frac{1}{2\pi^2}\varepsilon_{\mu\nu\sigma\tau}q^\sigma p^\tau, \tag{22b}$$

the term $T_{\mu\nu}$ being given by

$$T_{\mu\nu} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{tr}\left(\frac{\not{k}+m}{k^2-m^2}\gamma_\mu\frac{\not{k}-\not{p}+m}{(k-p)^2-m^2}\gamma_5\frac{\not{k}+\not{q}+m}{(k+q)^2-m^2}\gamma_\nu\right) + (p\leftrightarrow q, \mu\leftrightarrow\nu).$$

It is known that the anomaly can be shifted from the axial vector current to the vector current, or to a linear combination of these [8, 9]. Thus, by requiring that it be the vector current which is conserved, some of the freedom in the renormalization process is made use of.

The diagram of figure 3 is linearly divergent. If regularized by Taylor subtraction, in the spirit of BPHZ, it is given by

$$\begin{aligned}
\frac{1}{2}T_{\alpha\mu\nu} &= -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(1-t^1_{p,q}\right)\mathrm{tr}\left(\frac{\not{k}+m}{k^2-m^2}\gamma_\mu\frac{\not{k}-\not{p}+m}{(k-p)^2-m^2}\gamma_\alpha\gamma_5\frac{\not{k}+\not{q}+m}{(k+q)^2-m^2}\gamma_\nu\right)\\
&= -2\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left(1-t^1_{p,q}\right)\\
&\quad\times\frac{\mathrm{tr}[(\not{k}+m)\gamma_\mu(\not{k}-\not{p}+m)\gamma_\alpha\gamma_5(\not{k}+\not{q}+m)\gamma_\nu]}{[(k^2-m^2)(1-x-y)+((k-p)^2-m^2)x+((k+q)^2-m^2)y]^3}\\
&= -2\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left(1-t^1_{p,q}\right)\\
&\quad\times\frac{\mathrm{tr}[(\not{k}+m)\gamma_\mu(\not{k}-\not{p}+m)\gamma_\alpha\gamma_5(\not{k}+\not{q}+m)\gamma_\nu]}{[(k+(qy-px))^2-(qy-px)^2-m^2(1-x-y)+(q^2-m^2)y+(p^2-m^2)x]^3}.
\end{aligned}$$

A second term contributing to the chiral anomaly is obtained by interchanging $(p\leftrightarrow q)$, $(\mu\leftrightarrow\nu)$. By the symmetry of the integrands this second term yields the same result as the first so that the factor $1/2$ on the left-hand side can be dropped.

Following the rules of the modified BPHZ scheme described in section 3, one performs the substitution

$$\bar{k} = k-(qy-px)$$

so as to separate internal and external momenta, and to allow for separation of terms even and odd in the new integration variable $\bar{k}$. Indeed, only the even terms contribute to the integral. A straightforward calculation leads to the result

$$T_{\alpha\mu\nu} = T^{\log}_{\alpha\mu\nu} + T^{\mathrm{finite}}_{\alpha\mu\nu},$$

the logarithmically divergent term and the finite term being given by, respectively,

$$\begin{aligned}
T^{\log}_{\alpha\mu\nu} &= \frac{1}{2\pi^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\varepsilon_{\alpha\mu\nu\sigma}\{(3x-1)p^\sigma-(3y-1)q^\sigma\}\\
&\quad\times\ln\left(\frac{m^2}{m^2+(qy-px)^2-q^2y-p^2x}\right),\\
T^{\mathrm{finite}}_{\alpha\mu\nu} &= \frac{1}{2\pi^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\Bigg\{\varepsilon_{\alpha\mu\nu\sigma}((y-1)q^\sigma-(x-1)p^\sigma)\\
&\quad+\frac{\varepsilon_{\alpha\mu\nu\sigma}\{[(y-1)q^\sigma-(x-1)p^\sigma][(qy-px)^2-m^2]-yq^2p^\sigma+xp^2q^\sigma\}}{m^2+(qy-px)^2-q^2y-p^2x}\\
&\quad+2\frac{y\varepsilon_{\alpha\mu\sigma\tau}p^\sigma q^\tau[(y-1)q_\nu-xp_\nu]+x\varepsilon_{\alpha\nu\sigma\tau}q^\sigma p^\tau[(x-1)p_\mu-yq_\mu]}{m^2+(qy-px)^2-q^2y-p^2x}\Bigg\}.
\end{aligned}$$

In this example the well-known result from unmodified BPHZ, or from dimensional regularization [9], is obtained in a straightforward and technically simpler fashion. Replacing m^2 in the logarithmic integrand by an arbitrary parameter μ^2 does not change the total expression. Indeed, this replacement produces an additive term proportional to

$$\{(3x-1)p^\alpha - (3y-1)q^\alpha\} \ln\left(\frac{m^2}{\mu^2}\right),$$

which yields zero after integrating over the Feynman parameter y from 0 to $(1-x)$, and over x from 0 to 1.

For the sake of completeness, we verify the Ward identities (22*a*) and calculate the anomaly (22*b*). Straightforward calculation of the divergence $p^\mu T_{\alpha\mu\nu}$ leads to the result

$$\begin{aligned}
p^\mu T_{\alpha\mu\nu} = \frac{1}{2\pi^2}\varepsilon_{\alpha\mu\nu\sigma} p^\mu q^\sigma \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{m^2 + (qy-px)^2 - q^2 y - p^2 x} \\
\times \left[\left(-yx + 2yx^2 + x^3 - \frac{1}{2}x^2\right) p^2 + (yx^2 - y^2 x) qp \right. \\
\left. + \left(-y^3 - 2y^2 x + \frac{1}{2}y^2 + yx\right) q^2\right].
\end{aligned}$$

In the diagram of figure 3, the vector bosons at the lower vertices are identical so that $p^2 = q^2$ (and equal to zero in the case of external photons). With $q^2 = p^2$ the integrand is antisymmetric under exchange of x and y while the domain of integration is symmetric. Therefore, the integral vanishes and the first of the Ward identities (22*a*) holds true. The second Ward identity follows from the first by the symmetry $(p \leftrightarrow q)$, $(\mu \leftrightarrow \nu)$.

Regarding the divergence (22*b*) which contains the anomaly, we find for the first term on the right-hand side

$$2mT_{\mu\nu}(m) = -\frac{1}{\pi^2}\varepsilon_{\mu\nu\sigma\tau} q^\sigma p^\tau \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{m^2}{m^2 + (qy-px)^2 - q^2 y - p^2 x}.$$

The left-hand side of (22*b*) is calculated along the lines of the procedure described above. We find the result

$$\begin{aligned}
(p+q)^\alpha T_{\alpha\mu\nu} = -\frac{1}{\pi^2}\varepsilon_{\mu\nu\sigma\tau} q^\sigma p^\tau \\
\times \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \left\{\frac{m^2}{m^2 + (qy-px)^2 - q^2 y - p^2 x} - 1\right\},
\end{aligned}$$

and, upon comparison with the previous formula,

$$(p+q)^\alpha T_{\alpha\mu\nu} = 2mT_{\mu\nu}(m) + \frac{1}{2\pi^2}\varepsilon_{\mu\nu\sigma\tau} q^\sigma p^\tau,$$

which is, indeed, the anomalous identity (22*b*).

This example illustrates the advantage of the modified BPHZ procedure well, compared to original BPHZ renormalization or to dimensional renormalization, by its simplification of the momentum integral. Furthermore, the equivalence to EG regularization puts the modified procedure on solid ground. Compared to dimensional regularization, in particular, there is no need to introduce an analytic continuation of γ_5 to any other spacetime dimension than 4.

We note in passing that the chiral limit $m \to 0$, like in the usual BPHZ framework, requires a separate discussion. We do not treat this case in the present work.

4.3. Higher loop diagrams

The ‘sunrise’ diagram in the ϕ^4 model, cf figure 2, provides an instructive example for the comparison of BPHZ and EG regularizations. Being a diagram with two vertices it contains no EG subdiagrams at all. In the perspective of BPHZ, however, it contains three logarithmically divergent pure BPHZ subdiagrams. Thus, in the former case it is regularized in a single step by the Taylor subtraction with respect to the external momentum, while in the latter, one would have to invoke the forest formula for identifying the counter terms stemming from the three divergent subdiagrams. That is to say that the modified approach which, in essence, is a practicable version of EG is technically simpler, and furthermore it uses the fact that, in accord with Zimmermann’s theorem [10], the contributions from all pure BPHZ subprocesses cancel.

Regularizing the quadratically divergent diagram of figure 2 by Taylor subtraction of the integrand, one has

$$\Sigma(p) = \frac{g^2}{6(2\pi)^8}\int \mathrm{d}^4q \int \mathrm{d}^4k \big(1 - t_p^2\big)\frac{1}{[(p-k-q)^2 - m^2]}\frac{1}{k^2 - m^2}\frac{1}{q^2 - m^2}. \tag{23a}$$

One successively introduces Feynman parameters for the internal momenta k, q and $p-k-q$, and applies the necessary translations which decouple internal and external momenta. Details of this calculation are given in the appendix. The result is

$$\Sigma(p) = \frac{g^2}{6(4\pi)^4}\int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \frac{(1-2z)(1-2x)p^2}{(z-1)(1-z+z^2)} \times \ln\left(\frac{[z(1-z)(1-x)+x]m^2}{-xz(1-z)(1-x)p^2 + z(1-z)(1-x)m^2 + xm^2}\right). \tag{23b}$$

As in the previous examples, one replaces the parameter m^2 by an arbitrary parameter μ^2 but verifies that the result (23b) remains unchanged,

$$\Sigma^{(\mu)}(p) = \Sigma(p).$$

It is instructive to compare the result (23b) to a calculation of the sunrise diagram using dimensional regularization [12]. The result is

$$\Sigma_{\text{dim.reg}}(p) = \frac{g^2}{6(4\pi)^4}\int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \left\{\left(-\frac{(1-x)}{x}3m^2 + (1-x)p^2\right) \times \ln\left(\frac{[z(1-z)(1-x)+x]m^2}{-xz(1-z)(1-x)p^2 + z(1-z)(1-x)m^2 + xm^2}\right) + \frac{1}{2}p^2\right\}. \tag{23c}$$

The expressions (23b) and (23c) are both regularizations of the same scalar distribution. Furthermore, their Taylor expansion vanishes up to the order p^2, in the first case by construction and in the second case due to the additional term $p^2/2$. As this exhausts the remaining freedom in regularizing, one concludes that the two results are identical.

In order to make contact with the unmodified BPHZ procedure, we have checked by explicit calculation using the forest formula that, indeed, the three subdiagrams which are not EG subdiagrams cancel in the Taylor expansion and, hence, do not contribute to the regularization of the sunrise diagram. Thus, the modified BPHZ procedure is very close to pure EG regularization and avoids from the start irrelevant contributions from pure BPHZ subdiagrams to the regularized amplitudes, in agreement with the proof by Zimmermann [10].

These results are corroborated by case studies of EG regularization in higher orders. Among others we studied the four-point function of the ϕ^4 theory in dimension 4, at the level of two loops. The same model in dimension 6 provides an example which besides yielding divergent EG subdiagrams, also exhibits pure BPHZ subdiagrams. The contribution of the pure BPHZ subdiagrams, i.e. those which have no counterpart in EG, upon Taylor subtraction, are found to vanish, as expected. In the latter example we also studied three-loop contributions to the two-point function [6]. In all cases EG regularization, on one hand, and calculation following the forest formula restricted to EG subdiagrams, on the other hand, yield identical results.

5. Conclusions and outlook

The modified BPHZ procedure that we advocate in this paper combines the transparent concept of BPHZ regularization with the practical usefulness of dimensional regularization. In particular, the relevant integrals are easier to calculate than the corresponding ones within the original BPHZ method. Furthermore, the momentum-dependent logarithms always contain a reference mass which is identical with the typical mass parameter of the theory (the electron mass in the case of the examples from QED, the scalar mass in the ϕ^4 model). We showed, however, that rescaling is possible within the freedom allowed by the regularization process. In the examples with one loop, for instance, this allows us to introduce a new mass parameter which may be identified with the parameter of dimensional regularization. However, there is an essential difference here: while dimensional regularization requires the introduction of this parameter for (spacetime-)dimensional reasons, in our approach it is a manifestation of the general freedom within the process of regularization. This remark, in turn, justifies its appearance in the results of dimensional regularization.

The comparison of BPHZ regularization along the forest formula with the Epstein–Glaser construction confirms the expected significant simplification of explicit calculations in higher orders. In light of the different classifications of subdiagrams in the framework of BPHZ on one side, and in the Epstein–Glaser construction on the other, the summation over the subdiagrams contained in the forest formula is restricted to subdiagrams in the sense of Epstein–Glaser[1]. The modified procedure implies, in particular, that the combinatorics of higher order diagrams is described by the restricted forest formula which takes account exclusively of the class of EG subdiagrams. Thus, this method is as straightforward as, e.g. dimensional regularization, and has the virtue to rest on solid mathematical ground.

Appendix. Derivation of equation (23b)

The strategy for deriving (23*b*) goes as follows. A first Feynman parameter denoted by z is introduced for the k-integration. A subsequent translation by $k \mapsto \tilde{k} = k - (1-z)(p-q)$ then frees this inner momentum from mixed terms. Furthermore, we introduce the redundant operation $\left(1 - t_p^0\right)$, namely

$$\widetilde{\Sigma}(p) := \frac{6(2\pi)^8}{g^2}\Sigma(p) = \int \mathrm{d}^4 q\left(1 - t_p^2\right)\int_0^1 \mathrm{d}z \int \mathrm{d}^4 k\left(1 - t_p^0\right)$$
$$\times \frac{1}{\{(1-z)[(p-k-q)^2 - m^2] + z(k^2 - m^2)\}^2}\frac{1}{q^2 - m^2}$$
$$= \int \mathrm{d}^4 q\left(1 - t_p^2\right)\int_0^1 \mathrm{d}z \int \mathrm{d}^4 \tilde{k}\left(1 - t_p^0\right)\frac{1}{\{\tilde{k}^2 + z(1-z)(p-q)^2 - m^2\}^2}\frac{1}{q^2 - m^2}$$

[1] Of course, a certain choice of the standard momentum flow in the forest formula had to be made but the conclusion should be independent of that choice.

$$= 2\mathrm{i}\pi^2 \int \mathrm{d}^4q \big(1 - t_p^2\big) \int_0^1 \mathrm{d}z \int_0^\infty \mathrm{d}\rho \big(1 - t_p^0\big) \frac{\rho^3}{\{\rho^2 - z(1-z)(p-q)^2 + m^2\}^2} \frac{1}{q^2 - m^2}$$

$$= \mathrm{i}\pi^2 \int \mathrm{d}^4q \big(1 - t_p^2\big) \int_0^1 \mathrm{d}z \ln\left(\frac{m^2 - z(1-z)q^2}{m^2 - z(1-z)(p-q)^2}\right) \frac{1}{q^2 - m^2}. \tag{A.1}$$

Doing a partial integration in the integral over the parameter z, and introducing the abbreviation $\bar{m}^2 := m^2/(z(1-z))$, yields successively

$$\widetilde{\Sigma}(p) = -\mathrm{i}\pi^2 \int \mathrm{d}^4q \big(1 - t_p^2\big) \int_0^1 \mathrm{d}z \frac{z(1-2z)m^2(p^2 - 2pq)}{[m^2 - z(1-z)q^2][m^2 - z(1-z)(p-q)^2][q^2 - m^2]}$$

$$= -\mathrm{i}\pi^2 \int \mathrm{d}^4q \big(1 - t_p^2\big) \int_0^1 \mathrm{d}z \frac{zm^2}{z^2(1-z)^2} \frac{(1-2z)(p^2 - 2pq)}{[q^2 - \bar{m}^2][(p-q)^2 - \bar{m}^2][q^2 - m^2]}.$$

In evaluating the integration over the momentum q, one introduces two more Feynman parameters x and y so as to obtain

$$\widetilde{\Sigma}(p) = -2\mathrm{i}\pi^2 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \mathrm{d}^4q \big(1 - t_p^2\big) \frac{m^2}{z(1-z)^2}$$

$$\times \frac{(1-2z)(p^2 - 2pq)}{\{(1-x-y)(q^2 - \bar{m}^2) + x[(p-q)^2 - \bar{m}^2] + y(q^2 - m^2)\}^3}.$$

Translation of the integration variable $q \mapsto q + xp$ decouples the remaining internal momentum from the external momentum p so that one obtains

$$\widetilde{\Sigma}(p) = -2\mathrm{i}\pi^2 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \mathrm{d}^4q \big(1 - t_p^2\big)$$

$$\times \frac{m^2}{z(1-z)^2} \frac{(1-2z)(p^2 - 2pq - 2xp^2)}{\{q^2 - x^2p^2 - (1-y)\bar{m}^2 - ym^2 + xp^2\}^3}$$

$$= (-2\mathrm{i}\pi^2)^2 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^\infty \mathrm{d}r\, r^3 \frac{p^2m^2}{z(1-z)^2}$$

$$\times \big(1 - t_p^0\big) \frac{(1-2z)(1-2x)}{r^2 + x^2p^2 + (1-y)\bar{m}^2 + ym^2 - xp^2\}^3}$$

$$= \pi^4 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{m^2p^2}{z(1-z)^2}$$

$$\times \frac{(1-2z)(1-2x)x(x-1)p^2}{[x(x-1)p^2 + (1-y)\bar{m}^2 + ym^2][(1-y)\bar{m}^2 + ym^2]}$$

$$= \pi^4 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{(1-y)m^2 + z(1-z)ym^2}$$

$$\times \frac{z(1-2z)(1-2x)x(x-1)m^2p^4}{x(x-1)z(1-z)p^2 + (1-y)m^2 + z(1-z)ym^2}.$$

The integration over the parameter y yields

$$\widetilde{\Sigma}(p) = \pi^4 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x\ \frac{(1-2z)(1-2x)p^2}{(z-1)(1-z+z^2)}$$

$$\times \ln\left(\frac{(-x-(1-x)z(1-z))\big(x(1-x)z(1-z)p^2 - m^2\big)}{-xz(1-x)(1-z)p^2 + m^2 - (1-x)(1-z+z^2)m^2}\right).$$

Finally, one notices that the following integral vanishes, by the antisymmetry of the integrand under $x \longleftrightarrow (1-x)$,

$$\int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \frac{(1-2z)(1-2x)p^2}{(z-1)(1-z+z^2)} \ln\left(\frac{x(1-x)z(1-z)p^2 - m^2}{-m^2}\right) = 0.$$

Making use of this fact one obtains the result

$$\widetilde{\Sigma}(p) \equiv \frac{6(2\pi)^8}{g^2}\Sigma(p) = \pi^4 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \frac{(1-2z)(1-2x)p^2}{(z-1)(1-z+z^2)} \times \ln\left(\frac{[x+(1-x)z(1-z)]m^2}{-x(1-x)z(1-z)p^2+(1-x)z(1-z)m^2+xm^2}\right). \tag{A.2}$$

This is the result shown in equation (23*b*).

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